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PROGRESS REPORT.

The following gentlemen have been elected members of the Society :

(1) Mr. A. Y. G. Campbell, B.A., I.C.S., C.I.E., Collector of Chittoor, Chittoor ;

(2) Mr. V. Gopalaswami Aiyangar, B.A., Demonstrator in Mathematics, Central College, Bangalore (at concessional rate) ;

(3) Mr. R. Krishnaswami Aiyar, B.A., Professor of Mathematics, H. H. the Maharaja's College, Trivandram ;

(4) Mr. C. N. Ganapati, M.A., Acting Assistant Professor of Mathematics, Presidency Colloge, Madras, (at concessional rate) ;

(5) Mr. Lala Hukamchand, M.A., B.Sc., First Professor of Mathematics, Khalsa College, Amritsar.

2. Mr. John Maclean, M.A., B.Sc., Professor of Mathematics, Wilson College, Chowpatty, Bombay, has been elected a Life-member of the Society.

3. The Committee are highly gratified to note that in the New Year's Honours List, our Patron has been elevated to peerage as Lord Sydenham. Also, Mr. T. Sundara Row gets a C.I.S.O. and Mr. K. Balaram Iyer, a Rao Bahadurship. Our worthy Librarian, Principal R. P. Paranjpye, has been appointed an Honourable Member of the Council of His Excellency the Governor of Bombay.

4. Professor A. C. L. Wilkinson, M.A., F.R.A.S., and Mr. S. Narayana Aiyar, M.A., have duly taken their seats on the Committee in place of the two worthy retiring members : Professor B. Hanumanta Rau, B.A., C.I.S.O., and Mr. V. Ramaswami Aiyar, M.A. The warmest thanks of the Committee are due to these retiring gentlemen who have worked so zealously and assiduously for the progress of the Society.

5. The following books are received in the Library :—

(1) *La Logique Deductive*—(dans sa derniere de developpement) by Alessandro Padoa, Gauthier-Villars, Paris, 1912.

(2) *A Preparatory Arithmetic*—by C. Pendlebury, M.A., G. Bell and Sons, Limited, London, 1912, 1*s.*/6*d.*

(3) *Life and Scientific Work of P. G. Tait*—by Dr. C. G. Knott, D. Sc., (presented by Mr. B. J. Padshah through Mr. Balakram, M.A., I.C.S.).

(4) *Madras University Calendar for 1912*, Vols. I & II, and Examination papers Volume for 1911 ;

(5) *Panjab University Calendar for 1912 and 1913* ;

(6) *Bombay University Calendar for 1912-1913*, Vols. I and II.

POONA,
31st Jany. 1913.

D. D. KAPADIA,
Hony. Joint Secretary.

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On the Geometrical Interpretation of the Lorentz Transformation.

By A. Steichen, Dr. Phil, S. J., Professor of Physics,
St. Xavier's College, Bombay.

It is the object of this paper to discuss a simple connection between the two geometrical interpretations of the Lorentz-Transformation. This will be done in § 3. In order to make the paper independent of the literature published in various periodicals, not always available for the Indian student, a short summary of such portions of the subject as are necessary for my purpose is given in §§ 1, 2.

§. 1. *The Lorentz-Transformation.*

In modern physics, the much discussed principle of Relativity is becoming more and more important. It is based on the assumption that we never observe any absolute motion, and it explicitly states that, whatever may be the co-ordinates $x\ y\ z\ t$ which we choose, it must be possible to express the laws of nature by mathematical equations, the forms of which are independent of the choice of co-ordinates. There are in space ∞^3 different, but equally admissible, systems of co-ordinates moving relatively to each other with constant velocity[1].

Since the forms of the equations for physical laws are the same for the same law in any one of the ∞^3 admissible systems, there must be equations of transformation for passing from one system of coordinates to any other admissible system. These equations of transformation must be the same whatever may be the law under consideration. The physical law most suitable for finding these equations of transformation is the law of propagation of light in vacuo.

Let a luminous point be fixed at the origin of a system of coordinates (x,y,z,t). Suppose that at the moment $t=o$, a flash of light is emitted from the point A. The light-wave will be propagated in space in concentric spheres,

$$x^2+y^2+z^2-c^2t^2=0,\ c=\text{velocity of light},\ t<0.$$

Two other points B and C likewise fixed in the system and equally distant from A, will be reached simultaneously by the light-signal at a time t.

Next refer the same phenomenon to any other admissible system $(x'y'z't')$ which moves with a velocity v relatively to the first system of coordinates in the direction BC. We determine t' and the origin of $(x'y'z')$ so that the luminous point A is at the origin and emits the flash of light

[1] Lane, Das Relativitaets prinzip p. 33.

at the time $t'=0$. According to the principle of Relativity, the light-wave in the second system is likewise propagated in concentric spheres

$$x'^2+y'^2+z'^2-c^2t'^2=0.$$

But during the time t' the point A and the two points B and C, as seen from the moving system, have moved through a distance vt in the direction CB. Therefore B, C are not situated on the same sphere round the moving origin, the two points equally distant from A will not be reached simultaneously for an observer in the moving system by the wave of light. Hence it follows that two phenomena simultaneous in one system of coordinates are, as a rule, not so in any other system. This shows that in general time is measured differently in different systems and that each system must have its own time (local time, Ortszeit). For a fuller discussion of this important point we refer our readers to A, Einstein's paper[1].

In order to find the relations between $xyzt$ and $x'y'z't'$, we consider the differential equation for the propagation of light in vacuo.

Mathematically the problem may be stated thus:

Find the relations between $xyzt$ and $x'y'z't'$ which transform

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}=0, \quad \ldots \quad \ldots \quad (1)$$

into

$$\frac{\partial^2\phi}{\partial x'^2}+\frac{\partial^2\phi}{\partial y'^2}+\frac{\partial^2\phi}{\partial z'^2}-\frac{1}{C^2}\frac{\partial^2\phi}{\partial t'^2}=0. \quad \ldots \quad \ldots \quad (2)$$

As to the coordinates, we make the following assumptions: When $t=0$, let $x=y=z=x'=y'=z'=t'=0$; the x—axis parallel to the x'—axis and similarly the y—and z—axes respectively parallel to the y'—and z'—axes. Let the second system move without rotation with a velocity v parallel to the x—axis.

Mathematically these conditions are expressed by the equations:

$$x'=k\,(v)\,(x-vt),\ y'=m\,(v)\,y,\ z'=m(v)\,z,\ t'=l\,(v)\,t-n\,(v)x$$

where $k\,(v)$, $l\,(v)$, $m\,(v)$, $n\,(v)$ are all functions of v to be determined.

Substituting these values for $x'y'z't'$ in (2) we find, since this equation must be identical with equation (1),

$$k\,(v)=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}},\ m\,(v)=1,\ l\,(v)=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}},\ n\,(v)=l\,(v)\frac{v}{c^2}.$$

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[1] An.d. Physik. vol. 17 (1905) p. 891..., Jahrbuch d. Radivakto vol. 4. (1907) p. 411...See also Lane, Das Relativitaets prinzip p. 33...

Hence we get the following equations of transformation :

$$\left.\begin{aligned} x'&=\frac{x-vt}{\sqrt{1-\frac{v^2}{c^2}}},\ y'=y,\ z'=z,\ t'=\frac{t-\frac{v}{c^2}x}{\sqrt{1-\frac{v^2}{c^2}}} \\ x&=\frac{x'+vt'}{\sqrt{1-\frac{v^2}{c^2}}},\ y=y',\ z=z',\ t=\frac{t'+\frac{v}{c^2}x'}{\sqrt{1-\frac{v^2}{c^2}}}. \end{aligned}\right\} \quad \ldots \quad \ldots \quad (3)$$

These equations relate to the more important special case of the general transformation. They have been called by H. Poincare′ a *Lorentz-Transformation*, because they were first found by H. A. Lorentz, though in a very different way.

From these equations we draw the following important conclusions :

1. We exclude imaginary coordinates and infinite co-efficients ;

$\therefore \sqrt{1-\frac{v^2}{c^2}}$ must be real and greater than 0.

$\therefore 1 > \frac{v^2}{c^2}$ *i.e.*, the velocity of light is greater than any other velocity.

2. $$t=\frac{t'+\frac{v}{c^2}x}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \therefore dt=\frac{dt'}{\sqrt{1-\frac{v^2}{c^2}}},$$

i. e. a chronometer fixed in the system $(x'y'z't')$ and moving together with the system with a velocity v will go slower than when at rest.

3. $$x'=\frac{x-vt}{\sqrt{1-\frac{v^2}{c^2}}},$$

$\therefore$ the length of a rod placed parallel to the x—axis will be

$$s=x_1-x_2=(x_1'-x_2')\sqrt{1-\frac{v^2}{c^2}}=s_0\sqrt{1-\frac{v^2}{c^2}}$$

i. e., the moving rod undergoes a contraction in the direction of motion for an observer at rest. This is the contraction of the moving body assumed by Fitzgerald and H. A. Lorentz in order to explain Michelson's experiments.

It may be noted here that Lorentz and Einstein modified only our idea of time. But H. Minkowski changed also the generally accepted idea of space.

§. 2. *Minkowski's Geometrical interpretation of the Lorentz-Transformation.*

The late Prof. H. Minkowski has given two geometrical interpretations of the Lorentz-Transformation, which, at first sight, appear to differ very widely from each other.

(1) First interpretation.[1]

Let $x\ y\ z$ be the rectangular coordinates in space of a point and let t denote time. Time and space are always connected with each other; nobody has ever observed a space without a time, nor a time without a space. The combination of a point in space and a point of time $(x\ y\ z\ t)$, Minkowski calls a world-point (Weltpunkt). The variety (Mannigfaltigkeit) of all possible $(x\ y\ z\ t)$ points he calls the world (Welt). Consider a substance at a world-point $(x\ y\ z\ t)$ and suppose that to any dt there correspond changes $dx\ dy\ dz$ of the space coordinates, then we get a curve in the world, a world-line whose points are single-valued functions of t; where $\infty < t < \infty$.

This line represents the life of the substance. The whole world appears broken up into world-lines, and the physical laws are but the relations between these world lines.

Let c be a positive parameter, and consider

$$c^2t^2-x^2-y^2-z^2=1.$$

Geometrically this equation represents two spaces analagous to a hyperboloid of two sheets, separated by $t=0$. Consider the space in the region $t>0$. There are certain homogeneous linear transformations which change

$$c^2t^2-x^2-y^2-z^2=1,$$

into

$$c^2t'^2-x'^2-y'^2-z'^2=1.$$

Of these transformations we consider only the special case where y and z remain unchanged, *i.e.*, a transformation which affects only the xt—plane. The curve of intersection of the xt-plane with

$$c^2t^2-x^2-y^2-z^2=1,$$

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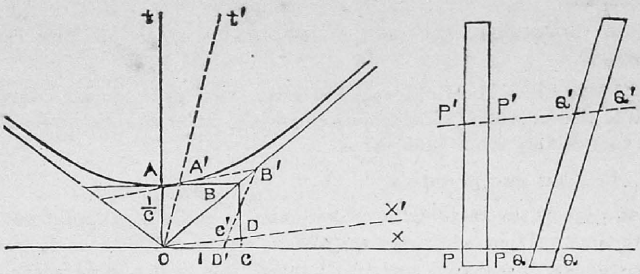
[1] Phys. Zeitsch. vol. 10 p. 104......, H. Minkowski, Raum and Zeit, (Leipzig, (1909) a reprint of the paper in the Phys. Zeitsch.

is the hyperbola

$$c^2t^2 - x^2 = 1.$$



Take any radius vector OA′; at A′ draw a tangent to the hyperbola meeting the asymptote at B′, from B′ draw a parallel to the radius-vector and from O a parallel to the tangent at A′. By taking OA′ as the new t'—axis and OC′ as x'—axis, the equation of the hyperbola will be

$$c^2t'^2 - x'^2 = 1,\ t' > 0,$$

if at the same time we put

$$\text{OC}' = 1 \text{ and } \text{OA}' = 1/c,$$

i.e., if we introduce with the new co-ordinates a new unit length.

When we have chosen our $x\ y\ z\ t$ coordinates, then to a point at rest there corresponds, as world-line, a straight line parallel to the t—axis; to a point moving with uniform velocity, a straight line inclined to the t—axis; to a point moving with variable velocity, a curved world-line. If at any world-point $(xyzt)$ the world-line is parallel to any radius-vector OA′ of the hyperboloid, then the substance may be considered at rest at this world-point when we choose OA′ as the new t—axis. Hence it follows that the substance at any world-point can always be considered at rest, provided we choose the axes of $x\ y\ z\ t$ properly.

Therefore

$$c^2dt^2 - dx^2 - dy^2 - dz^2$$

will always be positive, *i.e.*, c^2 always greater than $(ds/dt)^2 = v^2$, or

$$1 - v^2/c^2 > 0.$$

The hypothesis that the body undergoes a contraction in the direction of the motion, devised by H. A. Lorentz, finds a satisfactory explanation in the new way of considering space and time. We choose the co-ordinates so that the motion takes place in the xt plane alone.

The parallolegram PP′P′P, in the figure, parallel to the t—axis, represents the life of a body at rest, the parallelogram QQ′Q′Q that of a body moving with constant velocity. The bodies are supposed to have constant dimensions in space. When we change the axes of coordinates by taking the radius-vector parallel to QQ′ as the t'—axis and the corresponding x'-axis, then the second body appears to be at rest and the first body to move with a constant velocity, suppose that the first body, when considered at rest, has a length l parallel to the axis of x, *i.e.*

$$PP = l OC, \text{ where } OC = 1,$$

and that the second body when at rest has the same length l, then this means that

$$Q'Q' = l OC'$$

parallel to the x'—axis. The two bodies may represent two electrons, one at rest, the other moving with a constant velocity. In the original coordinates xt, the dimension of the second electron parallel to the x—axis, is QQ.

But $\quad Q'Q' = l \cdot OC'$

$\therefore \quad QQ = l \cdot OD'$

Also $\quad \dfrac{dx}{dt} = v$, for the second parallelogram.

$$\therefore \quad OD' = OC\sqrt{1-\frac{v^2}{c^2}}$$

$$\therefore \quad \frac{PP}{QQ} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

i.e., the moving electron appears, to an observer at rest, to undergo a contraction in the direction of its motion.

Lorentz calls the combination t' of x and t the local time of the electron moving with constant velocity.

Instead of taking t as one variable we may also take the product c as a variable[1]; then

$$c^2t^2 - x^2 = 1,$$

is a rectangular hyperbola, whose conjugate hyperbola is

$$c^2t^2 - x^2 = -1.$$

The common asymptotes are given by the equation

$$c^2t^2 - x^2 = 0.$$

[1] Lane, Das Relativitaets Prinzip p. 47.

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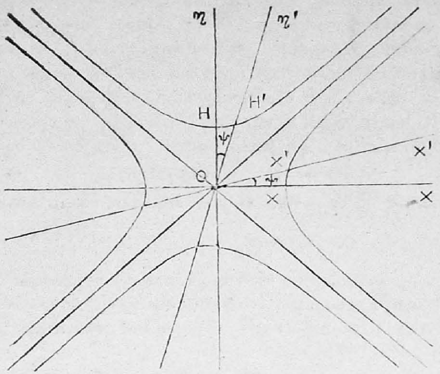
For the sake of shortness we simply write $ct=\eta$ and $ct'=\eta'$.



In the fig. the lines $O\eta'$ and Ox' are two conjugate diameters. From geometry we know that the equation

$$\eta^2-x^2=1,$$

will take the form

$$\eta'^2-x'^2=1,$$

when we take as the new axes of coordinates the two conjugate diameters $O\eta'$ and Ox' and as the new unit length $OH'=OX'=1$.

For our purpose it is important to find the relations between the new and the old coordinates in this transformation. The relations between the coordinates of any point are expressed by the two equations :

$$x'=a_1x+b_1\eta$$
$$\eta'=a_2x+b_2\eta.$$

In order to determine the four constants $a_1\ a_2\ b_1\ b_2$ four conditions must be given. These conditions obviously are in our case:

$$\left.\begin{array}{ll} OX'=x'_{X'}=1 & OH'=y'_{H'}=1 \\ y'_{X'}=0 & x'_{H'}=0 \end{array}\right\}\text{in the new coordinates.}$$

The coordinates of the same points H' and X' in the old system are readily found. The equation of the line Ox' is $\eta=x\tan\psi=x\beta$; that of the line $O\eta'$ is $x=\eta\tan\psi=\eta\beta$.

X' is the point of intersection of the lines Ox' with the hyperbola

$$x^2-\eta^2=1.$$

$$\therefore \qquad \eta_{X'} = \frac{1}{\sqrt{1-\beta^2}} \qquad x_{X'} = \frac{1}{\sqrt{1-\beta^2}}$$

H′ is the point of intersection of the line Oη' with the conjugate hyperbola $x^2 - \eta^2 = -1$.

$$\therefore \qquad \eta_{H'} = \frac{1}{\sqrt{1-\beta^2}} \qquad x_{H'} = \frac{\beta}{\sqrt{1-\beta^2}}$$

Hence we get the following values for the constants :

$$a_1 = \frac{1}{\sqrt{1-\beta^2}} \qquad a_2 = -\frac{\beta}{\sqrt{1-\beta^2}}$$

$$b_1 = -\frac{\beta}{\sqrt{1-\beta_2}} \qquad b_2 = \frac{1}{\sqrt{1-\beta^2}}$$

For x' and η' we find :

$$x' = \frac{x - \beta\eta}{\sqrt{1-\beta^2}} \qquad \eta' = \frac{\eta - \beta x}{\sqrt{1-\beta^2}}$$

For the Lorentz-Transformation in §. 1 we have found

$$x' = \frac{x - vt}{\sqrt{1-\frac{v^2}{c^2}}} \qquad t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1-\frac{v^2}{c^2}}}$$

substituting v/c for β, and ct, ct' for η and η', we see that we have the same relations between $x\,t$ and $x't'$, in either case.

Hence we conclude that the Lorentz-Transformation is equivalent to a change of co-ordinates connected with a change of unit length, the new axes of co-ordinates being conjugate diameters of the hyperbola $c^2t^2 - x^2 = 1$, and the new unit length being OX′.

Also $\qquad \psi < \frac{\pi}{4}. \quad \therefore \tan\psi = v/c < 1.$

$\therefore\ v < c =$ velocity of light.

(2) Second interpretation.[1]

Here we put $l = ict$, $\beta = v/c$.

$$\therefore \qquad x^2 + y^2 + z^2 - c^2t^2 = x^2 + y^2 + z^2 + l^2.$$

By the Lorentz—Transformation this expression will be changed into

$$x'^2 + y'^2 + z'^2 + l'^2.$$

The equations of the Lorentz—Transformation will take the form :

$$x' = \frac{x' + i\beta l}{\sqrt{1-\beta^2}},\ y' = y,\ z' = z,\ l' = \frac{l - i\beta x}{\sqrt{1-\beta^2}}.$$

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[1] Math. Annalen, vol. 68, p 472.. Lane, Das Relativitaets prinzip p. 53.

Let ϕ be an imaginary angle defined by the equations :

$$\cos\phi=\frac{1}{\sqrt{1-\beta^2}},\quad \sin\phi=\frac{i\beta}{\sqrt{1-\beta^2}}$$

$$\therefore \qquad \tan\phi=i\beta=i\,v/c=i\tan\psi$$

$$x'=x\cos\phi+l\sin\phi$$

$$l'=-x\sin\phi+l\cos\phi$$

$$x=x'\cos\phi-l'\sin\phi$$

$$l=x'\sin\phi+l'\cos\phi.$$

From the form of these equations it follows that the Lorentz-Transformation may be looked upon as a rotation of the system of coordinates in the xl—plane through an imaginary angle ϕ.

§. 3. Connection between the two geometrical interpretations.

In the second geometrical interpretation the angle ϕ is imaginary. Hence we put $\phi=iu$

$$\therefore \qquad \cos\phi=\cos(iu)=\cosh u=\frac{1}{\sqrt{1-\beta^2}}$$

$$\sin\phi=\sin(iu)=i\sinh u=\frac{i\beta}{\sqrt{1-\beta^2}}$$

$$\therefore \qquad \sinh u=\frac{\beta}{\sqrt{1-\beta^2}}$$

$$\therefore \qquad \tanh u=\beta=v/c=\tan\psi.$$

It is a well-known fact that the hyperbolic functions are connected with one branch of the rectangular hyperbola in the same way as the circular functions are with the circle.[2] Hence the imaginary angle ϕ leads us to a rectangular hyperbola in an $x\eta$—plane

$$\eta^2-x^2=a^2,$$

or $\qquad y^2-x^2=1$, by putting $a=1$.

From this hyperbola we get

$$x=\sinh u \qquad \eta=\cosh u.$$

In the first geometrical interpretation, we were led to the same rectangular hyperbola

$$\eta^2-x^2=1$$

where $\eta=ct$.

In this way the second geometrical interpretation leads directly to the first and vice versa.

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[1] Chryslal, Algebra II, p. 308.

Next we substitute cosh u for cos ϕ, and $i$sinh u for sin ϕ, in the equations for the Lorentz-Transformation. This gives :

$$x' = x \cosh u - ct \sinh u \qquad x = x' \cosh u + ct' \sinh u.$$
$$ct' = x \sinh u + ct \cosh u \qquad ct = x' \sinh u + ct' \cosh u.$$

These equations we compare with those for the rotation of coordinate axes through an angle Θ round the origin :

$$x' = x \cos \Theta + ct' \sin \Theta \qquad x = x' \operatorname{oos} \Theta - ct' \sin \Theta$$
$$ct' = -x \sin \Theta + c't' \cos \Theta \qquad ct = x' \sin \Theta + ct' \cos \Theta.$$

A comparison of the two sets of equations shows that the hyper. bolic functions play the same part in the Lorentz-Transformation as the circular functions in the rotation of axes. The difference in the sign in one place is due to the different behaviour of the axes in the two transformations. In the rotation of coordinates the two axes move in the same direction, they enclose a constant angle, in the Lorentz Transformation the two axes move in opposite directions, the angle between them is variable.

From geometrical considerations we readily understand why there is a change of unit length in the Lorentz-Transformation. For,

$$c^2t^2 - x^2 = 1,$$

changes into

$$c^2t'^2 - x'^2 = 1.$$

By introducing as new axes, two conjugate diameters of the rectangular hyperbola this curve has the equation

$$c^2t'^2 - x'^2 = a'^2.$$

where $a' > 1$. Hence in order to get the equation

$$c^2t' - x'^2 = 1,$$

we must increase the unit length, *i.e.*, for an observer himself in the system $(x't')$, the unit length is greater than for an observer outside the new system. This statement is equivalent to : for an observer outside the system $x't'$, the bodies in the moving system appear to undergo a contraction.

From the equations

$$\sinh u = \frac{\beta}{\sqrt{1-\beta^2}} \qquad \cosh u = \frac{1}{\sqrt{1-\beta^2}},$$

we get

$$\tanh u = \beta = v/c \qquad \therefore\ v = c \tanh u.$$

From the last equation we find immediately Einstein's theorem for the resultant of two velocities in the same direction. We put

$$v = c \tanh u = c \tanh (u_1 + u_2), \quad \text{where } u = u_1 + u_2.$$
$$= c\frac{\tanh u_1 + \tan hu_2}{1 + \tanh u_1 \tanh u_2} = \frac{v_1 + v_2}{1 + v_1v_2/c^2}.$$

This is Einstein's theorem[1]. If v_1 is the volccity with whieh the whole system moves parallel to the x—axis, v_2 the velocity parallel to the x'—axis of a body in the moving system, then v is the resultant velocity of the body as seen by an observer in the system (xt).

Also from $v=c \tanh u$, since $\tanh u$ is never greater than 1, we obtain v is never greater than c.

For the new unit length OX′ in the system $(x'\ t')$, measured in units of the original system $(x\ t)$, we find :

$$OX' = \sqrt{x^2_{X'} + \eta^2_{X'}} = \sqrt{\cosh^2 u + \sinh^2 u}.$$

In §. 1, we found that to a change of time dt in the system at rest there corresponds a change dt' in the moving system, and between these two elements of time we found the relation

$$dt = \frac{dt'}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{dt'}{\sqrt{1-\beta^2}}.$$

In our new notation this is written $dt = dt' \cosh u$.

A rod of length $s_0 = x_1' - x_2'$, in the moving system, viewed by an observer in the same system undergoes an apparent contraction if viewed by an observer at rest (§. 1). If s be the apparent length for the observer at rest, then

$$s = s_0 \sqrt{1-\frac{v^2}{c^2}}.$$

Hence we get

$$s_0 = s \frac{1}{\sqrt{1-\beta^2}} = s \cosh u.$$

These few examples may suffice. They show that the equations for the special Lorentz—Transformation and the equations derived from them, gain in elegance by introducing hyperbolic functions.

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[1] Annalen d. Physik, vol. 17 (1905), p 906.

SHORT NOTES.

The Harmonic Centre.

1. The following is an extension of the property of the harmonic centre [*vide* : Note on p. 96. J. I M. S., Vol. IV.]

Let $\rho_1, \rho_2, \dots \rho_n$ be the distances of a range defined by the equation $f(\rho)=\rho^n-p_1\rho^{n-1}+\dots\dots p_n=0$, the $(n-1)$ roots of

$$\frac{\lambda_1}{\rho-\rho_1}+\frac{\lambda_2}{\rho-\rho_2}+\dots\frac{\lambda_n}{\rho-\rho_n}=0 \quad \dots \quad (1)$$

are the *harmonic centres* of the range for the system of multiples $\lambda_1, \lambda_2 \dots\dots\dots \lambda_n$.

Evidently, if $A_1\ A_2 \dots\dots A_n$ be the range and $B_1B_2 \dots\dots\dots B_{n-1}$ their harmonic centres

$$\Sigma[\lambda_r/BA_r]=0,$$

for any B.

2. Interpreting the vector equation (1) we find that if $A_1\ A_2 \dots A_n$ be n points in a plane,

(i) forces invesely proportional to (λ_r/BA_r) acting along BA. are in equilibrium.

(ii) The product $(BA_1)^{\lambda_1}(BA_2)^{\lambda_2}\dots\dots$ is stationary for any B' and so on,

3. In the case of a triangle $(A_1A_2A_3)$, we have

$$\frac{\lambda_1}{\rho-\rho_1}+\frac{\lambda_2}{\rho-\rho_2}+\frac{\lambda_3}{\rho-\rho_3}=0,$$

leading to a vector quadratic. The roots of this determine two points (B_1B_2), such that

$$(\lambda_1+\lambda_2+\lambda_3)(\overline{OB_1}+\overline{OB_2})=[\overline{OA_1}(\lambda_2+\lambda_3)+\dots\dots],$$

$$(\lambda_1+\lambda_2+\lambda_3)\overline{OB_1}.\ \overline{OB_2}=\Sigma[\lambda_1\overline{OA_2}.\overline{OA_3}],$$

wherever o may be.

Hence, making O coincide with $A_1A_2A_3$ successively, we find

$$2(\lambda_1+\lambda_2+\lambda_3)\cdot A_1C=\overline{A_1A_2}(\lambda_1+\lambda_3)+\overline{A_1A_3}(\lambda_1+\lambda_2)$$

... ...

$$(\lambda_1+\lambda_2+\lambda_3)\overline{A_1B_1}.\ \overline{A_1B_2}=\lambda_1\ \overline{A_1A_2}.\ \overline{A_1A_3}$$

... ...

That is, B_1B_2 are the foci of the inscribed conic whose centre C has areal coordinates proportional to $(\lambda_2+\lambda_3)$ &c. In other words, B_1, B are the foci of the inconic whose areal equation is

$$[\Sigma(\lambda_1 x)^{\frac{1}{2}}=0.$$ [*vide* : Q. 416, J. I. M. S., Vol. IV. p. 200.]

4. The roots of

$$\sum\left[\frac{\lambda_1\lambda_2}{(\rho-\rho_1)(\rho-\rho_2)}\right]=0,$$

are $(n-2)$ in number, say $C_1\ C_2 \ldots C_{n-2}$; and are fixed relatively to the $A's$ and $B's$.

Thus in the above example

$$\overline{OC}\Sigma(\lambda_1\lambda_2)=\Sigma(OA_1.\lambda_2\lambda_3)$$

In other words, C is the mean centre of the $A's$ for the system of multiples $\lambda_2\lambda_3, \ldots\ldots$; that is, C is the point of concourse of the joins of the vertices to the points of contact of the in-conic.

5. Again, if the roots of

$$\sum\left(\frac{\rho-\rho_1}{\lambda_1}\right)=0, \sum\left[\frac{(\rho-\rho_1)(\rho-\rho_2)}{\lambda_1\lambda_2}\right]=0, \text{ etc.}$$

be defined as the *mean centres* of A_1A_2, of the 1st, 2nd, ... orders, for the system of multiples $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots$ ... ...it is clear that the harmonic centres of the rth order are the same as the mean centres of $(n-r)^{th}$ order.

30th August 1912. M. T. NARANIENGAR.

A Card Trick.

The following is an old and interesting mathematical card trick and perhaps some of your readers can explain why it works out.

From a pack of playing cards take out the jacks, queens, and kings, leaving thus 40 cards.

Now lay them face upwards in the following order :—

(Diamonds)	1	2	3	4	5	6	7	8	9	10
(Hearts)	10	1	2	3	4	5	6	7	8	9
(Spades)	9	10	1	2	3	4	5	6	7	8
(Clubs)	8	9	10	1	2	3	4	5	6	7

Next pick up the cards, one at a time, and lay them in a pile face downwards, beginning from the right upper corner (10 of diamonds) downwards to the 7 of clubs; follow with the next line to the left (9 of diamonds, 8 of hearts, &c., and so on to the last card at the left lower corner (8 of clubs). Now spread the cards again face downward, in 4 rows of 10 cards each, from left to right beginning with the card at the top of the pack (8 of clubs). The cards are now ready for performing the trick which consists in guesting the exact location of any of the cards, at the request of the spectators.

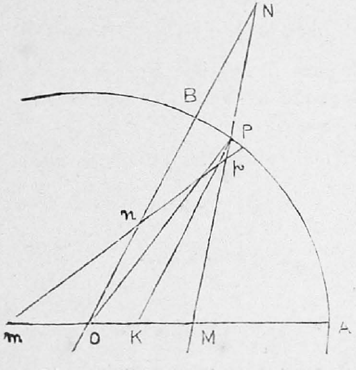
To work it out, multiply the number representing the value of the card called by 4 and add 3 for hearts, 6 for spades and 9 for clubs (nothing for diamonds). For instance, suppose the 2 of diamonds is called ; $2\times4=8$. The eighth card (counting from left to right first upper row) is the 2 of diamonds. The six of hearts would be found thus 6×4 equal to 24, plus 3 equal 27 ; the 27th card, *i.e.*, the 7th card in the third row should be the 6 of hearts, and so on.

When the product plus the number added exceeds 40, count the excess from the first upper row again ; for instance, the 9 of clubs should be 6×4 equal 36, plus 9 equal 45 (excess 5), the 5th card in the first row is the right card.

[From the Scientific American of 14-9-12.]

17th October, 1912. B. HANUMANTA RAU.

A Graphical Construction for the pth part of a given angle.



Let AOB$(=\omega)$ be the given angle ; draw the circle with centre O and radius $OA=OB=a$. Take on OA, $OM=\dfrac{a}{m}$; on OB, $ON=\dfrac{a}{n}$; join MN cutting the arc AB in P, so that $\angle BOP=\theta$.

The equation of MN is $xm+yn=a$; and the coordinates of P, lying on MN, are $x=OK=\dfrac{a\sin\theta}{\sin\omega}$, $y=PK=\dfrac{a\sin(\omega-\theta)}{\sin\omega}$. Hence, we have

$$m\sin\theta+n\sin(\omega-\theta)=\sin\omega \quad \dots\ (1)$$

giving θ in terms of ω.

As m and n are at our disposal, choose them so as to make the exdansion of $\sin\theta$ in terms of ω of the following form—

$$\sin\theta=\frac{\omega}{p}-\frac{\omega^3}{6\,p_3}+a_5\omega^5-a_7\omega^7,$$

where p is known, and a_5, a_7...have to be determind.

Supposing Θ to be acute, take the positive value of $\sqrt{(1-\sin^2\Theta)}$; we have, $\cos\Theta = 1-\frac{\omega^2}{2p^2}+\frac{\omega^4}{24p^4}-\frac{144a_5p^5-1}{144p^6}\,\omega^6$, working up to ω^7 only.

Substituting in equation (1) which gives, $\sin\Theta(m-n\cos\omega)+n\sin\omega\cos\Theta=\sin\omega$, we get

$$\left(\frac{\omega}{p}-\frac{\omega^3}{6p^3}+a_5\omega^5-a_7\omega^7\right)\left(m-n+\frac{n\omega^2}{2}-\frac{n\omega^4}{24}+\frac{n\omega^6}{720}\right)$$
$$+\left(n\omega-\frac{n\omega^3}{6}+\frac{n\omega^5}{120}-\frac{n\omega^7}{5040}\right)\left(1-\frac{\omega^4}{2p^2}+\frac{\omega^4}{24p^4}-\frac{144a_5p^5-1}{144p^6}\omega^6\right)$$
$$=\omega-\frac{\omega^3}{6}+\frac{\omega^5}{120}-\frac{\omega^7}{5040}\cdot\ \ldots\ldots$$

Identifying the coefficients of ω and ω^3, and solving for m and n, we get $(p\neq 1)$,

$$m=\frac{1-2p}{p-2},\ n=\frac{p+1}{p-2}.$$

If $p>2$, we see that m is negative ; hence by taking the $\left(\frac{2p-1}{p-2}\right)$ th part of OA in AO produced, and the $\left(\frac{p+1}{p-2}\right)$ th part of OB in OB, we find the angle BOP to be an approximate pth part of AOB. For instance, if $p=5$, $Om=\frac{1}{3}$ OA, $On=\frac{1}{2}$ OB ; and mn produced cuts the arc at p in such manner that $\angle\ \mathrm{BO}p=\frac{1}{5}\ \angle\ \mathrm{AOB}$ very nearly.

To estimate the error, we find a_5 by identifying the co-efficients of ω^5; this gives

$$\frac{n(-p^3+2p^2-2p+1)}{24p^4}+(m-n)a_5=\frac{1-n}{120},$$

whence
$$a_5=\frac{1}{120p}-\frac{(p^2-1)(p^2-p+1)}{72p^5}$$

It is found that for $p>2$, $a_5<\frac{1}{120\,p^5}$; and as terms in ω^7, ω^9 ......are very small in comparison with those in ω^5 for values of ω up to $\frac{1}{3}\pi$ $(p+3)$ we infer that Θ is slightly in defect of $\frac{\omega}{p}$.

$$\text{Hence }\ \frac{\omega}{p}-\Theta=\left(\sin\frac{\omega}{p}-\sin\Theta\right)\sec\frac{\omega}{p},\ \text{approximately}$$
$$=\left\{\frac{\omega^5}{120\,p^5}-\frac{\omega^5}{120p}+\frac{\omega^5(p^2-1)(p^2-p+1)}{72\,p^5}-a_7\omega^7\ldots\right\}$$
$$\times\left(1+\frac{\omega^2}{2p^2}+\ldots\right)\quad\ldots\ (3)$$
$$=\frac{\omega^5(p^2-1)(2p^2-5p+2)}{360p^5},\ \text{to the fifth power.}$$

If D is the number of *degrees* in the angle, the number of *seconds* in the error is seen to be

$$\frac{\pi^4 D^5}{3 \cdot 180^5} \cdot \frac{(p-1)(p+1)(p-2)(2p-1)}{360p^5};$$

which, as $\pi^4 < 100$, gives as an upper limit

$$\frac{D^5}{600 \cdot 18^4} \cdot \frac{(p-1)(p+1)(p-2)(2p-1)}{p^5}.$$

When $p=2$, the error is zero. Here m and n become infinite in the ratio -1; the exact bisection is obtained. I find that not only $\frac{1}{120p^5} - a_5$, but also $\frac{1}{5040p^7} - a_7$, &c., contain $p-2$ as a factor.

Taking the angle to be 52° 30′, it is found that

(i) when $p=3$, the error is $\frac{7 \cdot 35^4}{6 \cdot 36}$; or about 1′3″; here $OM = -\frac{1}{5}a$ $ON = \frac{1}{4}a$:

(ii) when $p=5$, the error is about 1′ 11″; $OM = -\frac{1}{3}a$, $ON = \frac{1}{2}a$:

and (iii) when $p=7$, the error is less than 1′ 11″; $OM = -\frac{5}{13}a$, $ON = \frac{5}{8}a$.

The terms in ω^7 omitted in the expansion of equation (1) and from (3) do not amount to more than $\frac{1'}{10}$ in any case.

It will be seen that for $p=3, 4, 5, \ldots$ the quantities $-\frac{1}{m}$ and $\frac{1}{n}$ form the series: $\frac{1}{5}, \frac{2}{7}, \frac{3}{9}, \frac{4}{11}, \frac{5}{13} \ldots\ldots, \frac{1}{4}, \frac{2}{5}, \frac{3}{6}, \frac{4}{7}, \frac{5}{8} \ldots\ldots$

K. J. SANJANA.

The Face of the Sky for March and April 1913.

Sidereal time at 8 p.m.

	March H.	M.	S.	*April.* H.	M.	S.
1	6	35	8	8	37	21
8	7	2	44	9	4	57
15	7	30	20	9	32	33
22	7	57	56	10	0	9
29	8	25	32	10	27	45

From this table the constellations visible during the evenings of March and April can be ascertained by a reference to their position as given in a star atlas.

Phases of the Moon.

	March.				April.			
	D.	H.	M.		D.	H.	M.	
New Moon ...	6	1	54	P.M. ...	5	1	27	A.M.
First Quarter ...	13	5	15	,, ...	11	10	7	P.M.
Full Moon ...	20	0	48	,, ...	18	11	24	,,
Last Quarter ...	28	5	34	A.M. ...	26	11	11	,,

The Sun.

enters the sign Aries or Vernal Equinox on March 21 at 10·48 a.m.

Eclipses.

A total lunar eclipse visible at Madras occurs on March 22nd.

First contact 3-32 p.m.
Middle ,, 5-18 ,,
Last ,, 7-6 ,,

A partial solar eclipse on April 6 is invisible at Madras.

Planets.

Mercury which was at superior conjunction on Feb. 13, attains its greatest elongation (18°19′E) on March 11 and becomes stationary on March 18. It is in conjunction with the moon on March 9 and April 5.

Venus which was at its greatest elongation (46°43′E) on Feb. 12, attains its greatest brilliancy on March 19 at 8-30 p.m.; and is stationary on April 3. It is in conjunction with the moon on March 12 and April 8 at 10-40 p.m.

Mars is in conjunction with the moon on March 5 and April 3.

Jupiter is in quadrature to the sun on April 7. It is in conjunction with the moon on March 30 and April 26.

Saturn is in conjunction with the moon on March 14 and April 10.

Uranus is in quadrature to the sun on April 28.

Neptune is stationary on April 4 and is in conjunction with the moon on April 4.

V. Ramesam.

REVIEWS.

Principia Mathematica: by Alfred North Whitehead, Sc.D., F.R.S., and Bertrand Russell, M.A., F.R.S., Vol. II. Cambridge University Press, 1912. Pp. xxxiv, 772: Price 30/—.

With this second volume of the great work of Dr. Whitehead and Mr. Russell, we get closer to the traditional subject of Arithmetic. This volume is written in symbolism to a far greater extent than was the first volume; for the first volume had to contain explanation of the symbolism used which are no longer necessary.

The second volume is divided into three parts, numbered III to V. Parts I and II were contained in the first volume, and, after a "Prefatory Statement of Symbolic Conventions," in which the various explanations which are required in applying the theory of types to cardinal arithmetic are brought together, Part III, on "Cardinal Arithmetic," begins. The first section is concerned with the definition and general logical properties of cardinal numbers. The definition of cardinal numbers is the one which was given by Frege in 1884 and 1893, and later, but independently, by Russell in 1903. The chief merits of this definition are, we read on p. 4, (1) that the formal properties which we expect cardinal numbers to have result from it; and (2) that unless we adopt this definition or some more complicated and practically equivalent definition, it is necessary to regard the cardinal number of a class as an indefinable. It is to be observed that the questions as to the existence of cardinal numbers now appear to be very much more complicated than they did in Mr. Russell's *Principles of Mathematics* of 1903 (Cf. pp. 12, 15 of the book under review). Also, it is not now remarked that one of the advantages of the definition of numbers as classes is that their existence can be proved.

Section B is on the operations of addition, multiplication, and exponentiation of cardinal numbers, in which it is to be noticed that the definitions and formal laws do not require any restriction to finite numbers. At the end of this section (pp. 171—184) there is a treatment of "greater and less" with cardinal numbers. It is to be noticed that we cannot define a "$\mu \geq \nu$" as "*either* $\mu > \nu$ *or* $\mu = \nu$"; because $\mu = \nu$ restricts μ and ν too much by requiring that they should be of the same type, and restricts them too little by not requiring that they should both be existent cardinals. But we read on p. 171 that "by the Schröder-Bernstein theorem, if μ is greater than or equal to ν, and ν is greater than or equal to μ, then $\mu = \nu$. This is misleading. We

are not given, in the Schröder-Bernstein theorem, that μ is greater than or equal to ν and ν is greater than or equal to μ. If we were, then $\mu=\nu$ would be a trivial deduction. What we *are* given is what the authors (perhaps misleadingly) symbolize by "$\mu\geq\nu$" and "$\nu\geq\mu$."

We must realize that extreme generality is aimed at by the authors. On p. 66 we read: "The treatment of addition, multiplication and exponentiation [given in this section] is guided by the desire to secure the greatest possible generality. In the first place, everything to be said generally about the arithmetical operations must apply equally to finite and infinite classes or cardinals. In the second place, we desire such definitions as shall allow the number of summands in a sum or of factors in a product to be infinite. In the third place, we wish to be able to add or multiply two numbers which are not necessarily of the same type. In the fourth place, we wish our definitions to be such that the sum of the cardinal numbers of two or more classes shall depend only upon the cardinal numbers of those classes, and shall be the same when the classes overlap as when they are mutually exclusive; with similar conditions for the product. The desire to obtain definitions fulfilling all these conditions leads to somewhat more complicated definitions than would otherwise be required; but in the outcome, the result is simpler than if we started with simpler definitions, since we avoid vexatious exceptions." If μ and ν are cardinals of assigned types, in the definition of their arithmetical sum $(\mu +_c \nu)$ various points should be noted. In the first place, it does not require that μ and ν should be of the same type; $\mu +_c \nu$ is *significant* whenever μ and ν are classes of classes. Thus it is not necessary for significance that μ and ν should be cardinals, though if they are not both cardinals, $\mu +_c \nu$ is the null class. Here we have an example of definition without hypotheses, which is one of the great advances in Frege's later works and is developed in these volumes. In this respect perhaps as much as in any other, there is a great and fundamental difference from Peano's work.

Section C deals with the theory of finite and infinite. This theory is rendered somewhat complicated by the fact that there are two different senses of 'finite,' which cannot (so far as is known) be identified without assuming the multiplicative axiom which has become particularly well-known to mathematicians by Zermelo's papers of 1904 and 1908, and by the work of Russell and other English mathematicians.

It is in this Part that the theory of types first becomes practically relevant. It will be found that contradictions concerning the maximum cardinal are solved by this theory.

Part IV deals with "Relation-Arithmetic": that is, a general kind of arithmetic of which ordinal arithmetic is a particular application; it is applicable to all relations, though its chief importance is in regard to such relations as generate series. The analogy with cardinal arithmetic is very close, and the reader will find that this Part is much facilitated by bearing the analogy in mind. The four sections are on: "Ordinal similarity and relation-numbers; Addition of relations, and the product of two relations; The principle of first differences, and the multiplication and exponentiation of relations; Arithmetic of relation-numbers."

Part V is on "Series". There is no use in distinguishing a series from its generating relation. A series is not a class, since it has a definite order, while a class has no order, but is capable of many orders (unless it contains only one term or none). The generating relation determines the order, and also the class of terms ordered, since this class is the field of the generating relation. Hence the generating relation completely determines the series, and may, for all mathematical purposes, be taken to *be* the series. Section A, "General theory of series", deals with the elementary properties of series, including maximum and minimum points, sequent points and limits; and Section B is "On Sections, Segments, stretches, and derivatives".

Section C "On convergence, and the limits of functions", which is likely to interest more mathematicians than other sections of the work, stands outside in the main developments of the book, and is concerned with convergence and the limits of functions and the definition of a continuous function. Its purpose is to show how these notions can be expressed, and many of their properties established, in a much more general way than is usually done, and without assuming that the arguments and values of the functions in turn are either numerical or numerically measurable. In the definitions usually given in treatises on analysis, it is assumed that both the arguments and the values of the function are numbers of some kind, generally real numbers, and limits are taken with respect to the order of magnitude. There is however, nothing essential in the definitions to demand so narrow a hypothesis. What is essential is that the arguments should be given as belonging to a series, and that the values should also be given as,

belonging to a series, which need not be the same series as that to which the arguments belong.

It will be remembered that Cantor, in his well-known paper of 1895, gave the generalizations of such notions as "limit," "continuity" of a variable, and various characteristics of point-aggregates to purely ordinal aggregates. Mr. Russell, in his book of 1903, emphasized this advance of Cantor's and remarked that such conceptions as those of the continuity, derivative, and definite integral of a function are not purely ordinal in their nature. It has since been discovered, apparently by various mathematicians, that the conception of the continuity of a function can be generalized into a purely ordinal conception. It is not the place here to emphasize the importance of these generalizations; we need only remark that certain theorems fail when we attempt to transfer them to series of which the ordinal type alone is given, thereby proving that these theorems depend essentially on the aggregates to which they apply being numerical.

In the third volume, the three remaining sections of the fifth part will be contained. These sections deal, respectively, with well-ordered series, with finite and infinite, and with "compact" ("everywhere dense" in Cantor's terminology) series.

The great thing which strikes us about the modern work of Dr. Whitehead and Mr. Russell is the comparatively complicated nature of the fundamental theorems of existence of mathematics. Ten years ago, everything seemed so simple: but then the paradoxes of logic quickly grew in importance, and matters had to be complicated to avoid the conclusion that mathematics is fundamentally false. The impression one gets from reading these volumes is like that one gets from looking through a microscope at a drop of water which is apparently clear to the naked eye. It is a most regrettable fact that the main body of mathematicians show such little interest in the logical foundations of their science. But it is still more regrettable that many of them think it possible to make statements of any value about these foundations without undertaking the labour of reading and reflecting on the splendid work of Dr. Whitehead and Mr. Russell.

PHILIP E. B. JOURDAIN.

SOLUTIONS.

Question 209.

(R. SRINIVASAN, M.A.) :—Any straight line drawn parallel to the y-axis to meet the Cissoid $y^2(a-x)=x^3$, cuts it in points the centres of curvature at which lie on a line.

Remarks by M. Satyanarayana, M. A.

The question is vague. The evolute of the cissoid is seen to be

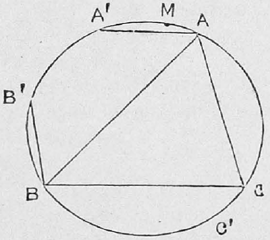
$$27y^4+288a^2y^2+512a^3x=0.$$

Question 300.

(ZERO) :—If A, B, C be three points on a circle, find a point M on the circle such that MA, MB, MC, are equal to the sides of another triangle inscribed in the same circle.

Solution by R. D. Karve, M.A.

Draw AA′, BB′, CC′ parallel to BC, CA, AB respectively. Then A′, B′, C′ are points on the circle satisfying the requisite conditions. For, A′B being equal to AC, the triangle A′AC has its sides equal to A′A, A′B, A′C. Similarly, in the triangle AA′B, AB is equal to A′C.



Remarks by 'Zero'.

Suppose the arc AM is θ. Then the sum of the arcs AM, BM, CM is $(3\theta+2A+4B)$. The condition that MA, MB, MC should be equal to the sides of an inscribed triangle is easily seen to be equivalent to

$$\text{arcs }(AM \pm BM \pm CM) = \text{a multiple of } 2\pi.$$

Taking two *plus* signs and one *minus* sign, we get the points A′, B′, C′ as determined above. If all the signs are *plus*, we get three additional solutions, *viz.*, the nearer points of trisection of the arcs AA′, BB′, CC′, which determine the vertices of an equilateral triangle. Thus, if M be the nearer point of trisection of the arc AA′, the chords MA, MB, MC are equal to the sides MN, MB, NB of △ M N B, where N is the other point of trisection of the arc AA′.

Question 302(*a*).

(S. NARAYANAN, B.A., L.T.) :—If δ be the circle described on the base BC of a triangle ABC as diameter, and σ the ex-cosine circle corresponding to A, prove geometrically that (1) the pedal triangle of any point on σ is right-angled, and (2) the isogonal transformation of σ is δ.

Solution by R. Tata, M. A.

Denote the centre of the excosine ⊙ by K_1 then AK_1 is the symmedian through A. Therefore the circle cuts the sides AB, AC again at the extremities of a diameter say B′C′. If now P be any point on this circle whose pedal triangle is DEF, we have the angle

	PDF = PBF,	from the cyclic quad.		PDBF
	= PC′B′	"	"	PC′B′B,
and	PDE = PCE	"	"	PDCE
	= PB′C′	"	"	PCC′B′.
∴	FDE = PC′B′ + PB′C′			
	= a right angle, since B′C′ is a diameter of σ.			

Produce BP to meet the circumcircle ABC in P′; draw CQ isogonal conjugate to CP with respect to the angle C, cutting the circle δ in Q join B Q.

It is easily seen that BPC = 90° + A.

∴ PCP′ is a right angle.

Inclination of Simson line of P′ to BC

= complement of ABP′ = PBC′ = PCA.

Inclination of BQ to BC is QBC = complement of QCB.

But PCA = QCB, as CP, CQ are isogonal.

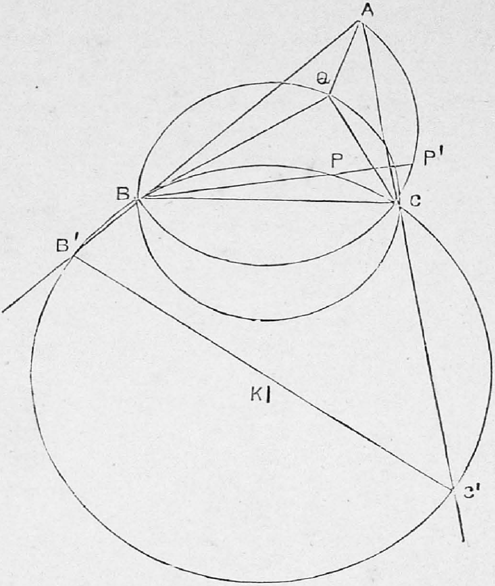
∴ Simson line of P′ and BQ are perpendicular.

But the simson line of P′ is perpendicular to the isogonal conjugate of BP′, by a well-known theorem.

∴ BQ, BP are isogonal conjugates for the angle B.

∴ P, Q are isogonal conjugates for the triangle ABC.

Thus the isogonal transformation of σ is δ.

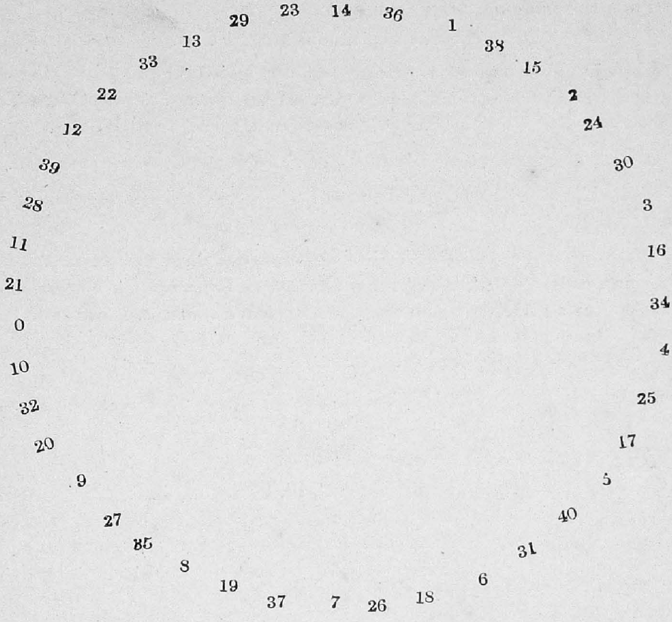


Question 314.

(Selected):—41 men are arranged in a ring which closes up its ranks as every third individual is picked out beginning at a particular position. If a certain named one should be left ultimately, find his position relative to the first one counted.

Solution by N. Sankara Aiyar and N. B. Pendse.

The accompanying figure shows that the given man should be the 31st in the order counted the first man being the one with whom the counting begins. The figures show the order in which the men are picked out.



[The general rule may be stated thus:—Let n men be arranged as described. Begin anywhere and go continuously round picking out each mth man, until r only are left. Suppose one of these is the man who originally occupied th p^{th} place. Then, if there had been $(n+1)$ men to begin with, one of the r left would have been the $(p+m)^{th}$, or, if $(p+m) > (n+1)$, the $(p+m-n-1)^{th}$. Therefore, if $f(n)$ be the number required $f(n+1) - f(n) = m$, &c.

Vide: C. G. Knott's *Life of P. G. Tait*, p. 118.]

Question 354.

(S. NARAYAN, B.A., L.T.) :—If P, Q, R be the points of trisection (next to the vertices) of the altitudes of a triangle ABC whose symmedian point is K, shew that the polar conic of K with respect to PQR is the locus of the centres of mean position of the feet of perpendiculars from any point on the circumcircle of ABC on its sides.

Solution by M. Bhimasena Rao.

Let DEF be the pedal triangle of any point O and O′ be the mean centre of D, E, F for multiples p, q, r. If the normal *c's* of O are l, m, n the normal *c's* of D, E, F are 0, $(m+l\cos C)$, $(n+l\cos B)$, etc.

∴ The normal *c's* of O′ are $l(q+r)+qm\cos C.+rn\cos B$, etc.

Let A′, B′, C′ be the positions of O′ when O coincides with A, B, C respectively.

The normal *c's* of A′, B′, C′ are $q+r$, $p\cos C.$, $p\cos B$; etc., so that the mean centre of A′, B′, C′ for multiples al, bm, cn is O′. Since O is the mean centre of ABC for the same multiples al, bm, cn, we see that O and O′ are corresponding points of ABC and A′B′C′.

The above shows that when O moves on any curve, O′ moves on the corresponding curve of A′B′C′, which is of course of the same degree as the locus of O.

In the question, O moves on the circumcircle of ABC and the multiples p, q_1 r are equal. The points A′, B′, C′ are P, Q, R. The locus of O′ is that circumconic of PQR which corresponds to the circumcircle of ABC. P, Q, R are corresponding poiuts of directly similar figures described on the altitudes of ABC. Hence the double point of ABC and PQR is K, which is the director point of the similar figures. (See, §3 of my Note on the Double Point, J.I.M.S., Vol. IV, p. 20). Since the circumcircle of ABC is the polar conic of K with respect to, ABC, the locus given in the question is the polar conic of K with respect to PQR.

Question 368.

(M. BHIMASENA RAU) :—Through the median point of a triangle a line is drawn cutting the circumcircle in P and Q. Show that the polars of P and Q with respect to the triangle are, respectively, perpendicular to the pedal lines of Q and P.

Solution by R. Tata, M.A.

Let P, Q be the points whose normal coordinates are $(\alpha_1\beta_1\gamma_1)$ $(\alpha_2\beta_2\gamma_2)$; since they are on the circumcircle, the equation to PQ is

$$\frac{a\alpha}{\alpha_1\alpha_2}+\frac{b\beta}{\beta_1\beta_2}+\frac{c\gamma}{\gamma_1\gamma_2}=0.$$

Since PQ passes through the centroid $\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$, we have

$$\frac{1}{\alpha_1\alpha_2}+\frac{1}{\beta_1\beta_2}+\frac{1}{\gamma_1\gamma_2}=0 \quad \dots \quad \dots \quad (1)$$

Since P is on the circumcircle, we have

$$\frac{a}{\alpha_1}+\frac{b}{\beta_1}+\frac{c}{\gamma_1}=0. \quad \dots \quad \dots \quad (2)$$

From (1) and (2) $\quad a\alpha_2=\frac{(b\gamma_1+c\beta_1)\beta_2\gamma_2}{\beta_1\beta_2+\gamma_1\gamma_2} \quad \dots \quad \dots \quad (3)$

The isogonal conjugate of AP is evidently

$$\beta\beta_1-\gamma\gamma_1=0 \quad \dots \quad \dots \quad (4)$$

The trilinear polar of Q is

$$\frac{\alpha}{\alpha_2}+\frac{\beta}{\beta_2}+\frac{\gamma}{\gamma_2}=0 \quad \dots \quad \dots \quad (5)$$

And (3) and (4) will be parallel provided

$$\begin{vmatrix} a & b & c \\ o & \beta_1 & -\gamma_1 \\ \frac{1}{\alpha_2} & \frac{1}{\beta_3} & \frac{1}{\gamma_2} \end{vmatrix}=0$$

i.e., $$\frac{\beta_1\beta_2+\gamma_1\gamma_2}{\beta_2\gamma_2}=\frac{b\gamma_1+c\beta_1}{a\alpha_2},$$

which has been proved in equation (3)

Thus the isogonal conjugate of AP is parallel to the trilinear polar of Q ; but it is known that the isogonal conjugate of AP is perpendicular to the pedal line of P.

Therefore the polar of Q is $\perp$ pedal line of P.

Question 369.

A. C. L. WILKINSON, M. A., F. R. A. S.) :—If a surface be inverted from any point O, and if ρ_1, ρ_2 ; ρ_1', ρ_2'' be the principal radii of curvature at any two corresponding points P, P′, shew that

$$OP'\left(\frac{1}{\rho_1'}-\frac{1}{\rho_2'}\right)=OP\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right).$$

Solution by M. R. Sadasiva Iyer.

In a plane curve and its reciprocal, if r and r' represent corresponding vectors, and ρ, ρ' the radii of curvature, it is easily seen that

$$\frac{r}{\rho}+\frac{r'}{\rho'}=2\sin\phi\ ;\ \text{since}\ \frac{r}{\rho}=\frac{dp}{dr}=\frac{d(r\sin\phi)}{dr}=\sin\phi+r\cos\phi\frac{d\phi}{dr},$$

$$\frac{r'}{\rho'}=\frac{dr'\sin\phi}{dr'}=\sin\phi+r'\cos\phi\frac{d\phi}{dr'}.$$

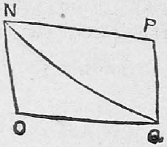
Let P be the point on the surface, O the centre of inversion, PQ a small arc of line of curvature. If the inclination of plane OPQ to the normal be α_1, $\rho=\rho_1\cos\alpha_1$, where ρ_1 is the principal radius of curvature.

Hence $$\frac{r}{\rho_1\cos\alpha_1}+\frac{r'}{\rho_1'\cos\alpha_1'}=2\sin\phi.$$

But $\alpha_1'=\pi-\alpha_1$; since a circle of curvature inverts into a circle of curvature.

$$\therefore\quad \frac{r}{\rho_1}-\frac{r'}{\rho_1'}=2\sin\phi\cos\alpha_1.$$

With P as centre, describe a unit sphere, N,O and Q being the points of intersection of the normal, radius vector and PQ with the sphere.



Then $$\cos NO=\sin\phi\cos\alpha_1.$$

$$\therefore\quad \frac{r}{\rho_1}-\frac{r'}{\rho_1'}=2\cos NO\ ;\ \text{similarly}\ \frac{r}{\rho_2}-\frac{r'}{\rho'_2}=2\cos NO.$$

$$\therefore\quad r\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)=r'\left(\frac{1}{\rho_1'}-\frac{1}{\rho_2'}\right)$$

Question 384.

(D. D. KAPADIA, M.A., B.SC.) :—When the expression $f(x)\{\phi(x)\}^{-2}$ is resolved into partial fractions, where $\phi(x)$ is the product of simple factors show that the necessary and sufficient condition that no terms of the type $A/(x-\alpha)$ should occur is that

$$f'(x)\phi'(x)-f(x)\phi''\phi''(x)$$

should be devisible by $\phi(x)$

Solution by J. C. Swaminarayan M.A.

In general,

$$\frac{f(x)}{\{\phi(x)\}^2}=\Sigma\frac{A}{x-\alpha}+\Sigma\frac{B}{(x-\alpha)^2}$$

Put $x-\alpha=y$, then

$$\frac{f(\alpha+y)}{[\phi(\alpha+y)]^2}=\frac{f(\alpha)+yf'(\alpha)+\ldots}{y^2[\phi'(\alpha)+y\phi''(\alpha)/\underline{|2}+\ldots]^2}=\frac{A}{y}+\frac{B}{y^2}+\ldots$$

Multiply by y^2, and put $\lambda=0$,

$$B=f(\alpha)/[\phi'(\alpha)]^2.$$

Differentiating with respect to y and putting $y=0$, we have

$$A=\frac{f'(\alpha)\phi'(\alpha)-f(\alpha)\phi''(\alpha)}{\{\phi'(\alpha)\}^3}.$$

Thus the necessary and sufficient condition that A should vanish is

$$f'(\alpha)\phi'(\alpha)-f(\alpha)\phi''(\alpha)=0.$$

Hence α which represents any one of the roots of the equation $\phi(x)=0$, is also a root of the equation

$$f'(x)\phi'(x)-f(x)\phi''(x)=0.$$

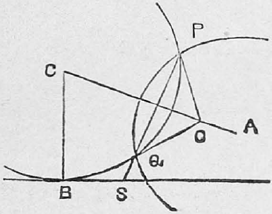
That is $f'(x)\phi'(x)-f(x)\phi''(x)$ should be divisible by $\phi(x)$.

Question 390.

(M. V. Arunachala Sastry, M.A.,) :—A variable circle C touches a given straight line at a given point and cuts a given circle in two points P, Q. Find the locus of the centres of circles cutting the circle C orthogonally at P,Q.

Solution (1) by N. B. Pendse, (2) by T. P. Trivedi, M.A., LL.B., and several others.

(1) Let A be the centre of the given circle, and C that of the variable circle touching the given line BS at B. Also, let PQ meet BS at S. Then S is a fixed point, since SB is equal to the tangent from S to the fixed circle.



The centre of the circle cutting C orthogonally at P,Q is the pole O of PQ. Therefore, the polar of S passes through O and is coincident with BO. That is, CS and BO are at right angles.

Now, if $O_1O_2O_3O_4$ be four points corresponding to $C_1C_2C_3C_4$, we have $A(O_1O_2O_3O_4)=A(C_1C_2C_3C_4)=S(C_1C_2C_3C_4)=B(O_1O_2O_3O_4)$.

Hence, the locus of O is a conic through A,B.

(2) Take the point as the origin and the given straight line as the axis of x. Then the equation of the variable circle is

$$x^2+y^2-2ry=0 \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

where r goes on changing.

Let the given circle be

$$(x-h^2)+(y-k)^2=a^2 \ldots \quad \ldots \quad \ldots \quad (2)$$

Any circle through the points of intersection of (1) and (2) is

$$(x^2+y^2)(1+\lambda)-2hx-2y(k+r\lambda)+h^2+k^2-a^2=0 \quad \ldots \quad (3)$$

Its centre (x_1, y_1) is given by

$$x_1 = \frac{h}{1+\lambda} = y_1 = \frac{k+r\lambda}{1+\lambda} \quad \dots \quad \dots \quad \dots \quad \dots \quad (4)$$

The square of its radius

$$= \frac{h^2}{(1+\lambda)^2} + \frac{(k+r\lambda)^5}{(1+\lambda)^2} - \frac{(h^2+k^2-a^2)}{1+\lambda}$$

If (1) and (3) are orthogonal, we have

$$\frac{h^2}{(1+\lambda)^2} + \left(\frac{k+r\lambda}{1+\lambda} - r\right)^2 = r^2 + \frac{h^2}{(1+\lambda)^2} + \frac{(k+r\lambda)^2}{(1+\lambda)^2} - \frac{(h^2+k^2-a^2)}{1+\lambda}$$

or $$2r\,(k+r\lambda) = h^2+k^2-a^2 \quad \dots \quad \dots \quad (5)$$

Eliminating r and λ between (4) and (5) we see that the locus of (x_1, y_1) is given by

$$2\,hy\,(yh-xk) = x(h-x)(h^2+k^2-a^2)$$

or $$2h^2y^2 + x^2(h^2+k^2-a^2) - 2hkxy - xh(h^2+k^2-a^2) = 0,$$

i.e., a conic passing through the origin and the point (h, k), or passing through the fixed point and the centre of the given circle.

Question 391.

(D. D. KAPADIA, M.A., B.Sc.) :—Prove that the sum of the reciprocals of the co-efficients in the expansion of $(1-x)^{2n}$ is $2-1/(n+1)$ where n is a positive integer.

Solution by V. K Aravamudan, B.A., T. P. Trivedi, M.A., LL.B., and G. Ramachandran.

We know

$$-\frac{a}{b} + \frac{a(a+x)}{b(b-x)} - \frac{a(a+x)(a+2x)}{b(b-x)(b-2x)} + \dots$$

$$= \frac{1}{a+x+b}\left\{\frac{a(a+x)\dots(a+nx)}{(-1)^n b(b-x)\dots(b-\overline{n-1}x)} - a\right\}$$

(*Vide*: C. Smith's *Alg.*, p. 411.)

Now the series in question

$$= 1 + \left\{-\frac{1}{2n} + \frac{1.2}{2n(2n-1)} - \frac{1\cdot2\cdot3}{2n\cdot\overline{2n-1}\cdot\overline{2n-2}} + \dots\text{to } 2n \text{ terms}\right\}$$

and is got by putting $a=1$, $b=2n$; $x=1$.

Hence it is equal to

$$1 + \frac{1}{2n+2}\left\{\frac{1\cdot2\cdot3 \quad \dots 2n+1}{2n\cdot\overline{2n-1}\dots\ 2\cdot1} - 1\right\} = 1 + \frac{n}{n+1} = 2 - \frac{1}{n+1}$$

Additional Solutions by J. O. Swaminarayan, M.A., and N. P. Pandya.

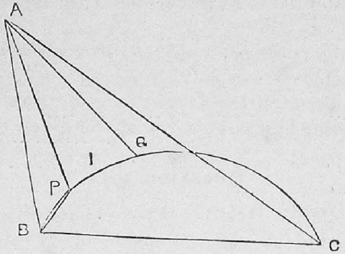
Question 393.

(T. RAJARAMA RAO) :—A is the vertex of a triangle on a given base BC, having a given vertical angle. Find the locus of the isogonal conjugates of the triangle ABC that are equidistant from A.

Solution by N. B. Pendse and V. V. Satyanarayana.

Let P,Q be the two points for the triangle ABC in one particular position and I the incentre.

Then BI is the bisector of $\angle PBQ$ and AI is the perpendicular bisector of PQ. Therefore PQIB are concyclic and similarly PQIC are concyclic.



$\therefore$ P,Q are on the circle BIC.

$\therefore$ $BPC = BQC = BIC = \pi/2 + A/2$.

As the angle A is given, on BC describe a segment BIC of a circle containing an angle equal to $(\pi/2+A/2)$. Then BIC is the required locus of P and Q.

[Also, see p. 175 of J. I. M. S, Vol. IV.]

Question 396.

(K. J. SANJANA, AND T. P. TRIVEDI) :—Prove the following identities :—

(i) $(n+1)n\sin\theta + n(n-1)\sin 2\theta + (n-1)(n-2)\sin 3\theta + \dots + 2\cdot 1 \sin n\theta$

$$= \frac{1}{2}(n+1)(n+2)\cot\frac{\theta}{2} - \frac{1}{2}\operatorname{cosec}^2\frac{\theta}{2}\sin\frac{n+1}{2}\theta\sin\frac{n+2}{2}\theta;$$

(ii) $n(n+1)\sin n\theta + n(n-1)\sin(n-1)\theta + \dots + 2.1.\sin\theta$

$$= \frac{1}{4}\operatorname{cosec}^2\frac{\theta}{2}\left\{(n+1)(n+2)\sin n\theta - (n+1)n\sin(n+1)\theta\right\}$$

$$-\frac{1}{2}\operatorname{cosec}^3\frac{\theta}{2}\sin\frac{n\theta}{2}\sin\frac{n+1}{2}\theta;$$

(iii) $(n-1)^3 \sin\theta+(n-2)^3 \sin 2\theta+\ldots+1^3 \sin(n-1)\theta$

$$=\frac{1}{4}\operatorname{cosec}^2\frac{\theta}{2}\left\{n^3 \sin n\theta-\sin n\theta\right\}$$

$$-\frac{3}{8}\operatorname{cosec}^4\frac{\theta}{2}\left\{n\sin\theta-\sin n\theta\right\}$$

(iv) $(n-1)^4 \cos\theta+(n-2)^4 \cos 2\theta+\ldots+1^4 \cos(n-1)\theta$

$$=-n^4+n^3 \operatorname{cosec}^2\frac{\theta}{2}-\tfrac{1}{2}n \operatorname{cosec}^2\frac{\theta}{2}\left(3\operatorname{cosec}^2\frac{\theta}{2}-1\right)$$

$$+\frac{1}{8}\sin n\theta \sin\theta \operatorname{cosec}^4\frac{\theta}{2}\left(3\operatorname{cosec}^2\frac{\theta}{2}-1\right).$$

Solution by N. B. Pendse, G. Ramachandran, A. A. Krishnaswami Iyengar N. P. Pandya, and others.

Let $S\equiv\sin\theta+\sin 2\theta+\ldots\ldots\ldots\sin n\theta$

$S_1\equiv(n-1)\sin\theta+(n-2)\sin 2\theta+\ldots\ldots\sin(n-1)\theta,$

$S_2\equiv(n-1)^2\sin\theta+(n-2)^2\sin 2\theta+\ldots\sin(n-1)\theta;$

... ...

and $C\equiv\cos\theta+\cos 2\theta+\ldots\ldots\ldots\ldots\cos n\theta,$

$C_1\equiv(n-1)\cos\theta+(n-2)\cos 2\theta+\ldots\ldots\cos(n-1)\theta,$

... ...

The values of S and C are well known. Those of S_1, $S_2 \ldots C_1$, C_2 are found as follows:

Differentiate C with respect to θ and subtract from S_1. We have $S_1-C'=nS$, the accent devoting differentiation with respect to θ.

$\therefore$ $S_1=nS+C'$.

Similarly $S_2=nS_1+C_1'$

... ...

Also, $C_1=nC-S'$,

$C_2=nC_1-S_1',\ldots$

... ...

Now, suppose it is required to find the value of

$$\Sigma\equiv a_{n-1}\sin\theta+a_{n-2}\text{S in }2\theta+\quad\ldots\quad a_0 \sin n\theta,$$

where a_r is a known function of r, say $f(r)$.

$$\Sigma\equiv f(o).S+f'(o)S_1+\frac{f''(o)}{|2}S_2+\ldots\qquad\text{, as is easily seen.}$$

Since S, S_1, S_2, ...are all known, Σ is readily obtained.

The details are omitted as they can be easily supplied by the reader

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Question 400.

(D. D. KAPADIA, M.A., B.Sc.) :—Prove that if $|x|<1$,

$$\frac{x}{(1+x)^2}+\frac{x^2}{(1+x^2)^2}+\frac{x^3}{(1+x^3)^2}+\ldots\ldots\ldots\ldots$$

$$=\frac{x}{1-x}-\frac{2x^2}{1-x^2}+\frac{3x^3}{1-x^3}-\ldots\ldots=\sum_1^{\infty}\psi(\mu)x^{\mu},$$

where $\psi(\mu)$ denotes the excess of the odd over the even divisors of μ (1 and μ being reckoned as divisors).

Solution by V. B. Naik, T. P. Trivedi and J. C. Swaminarayan.

Since $|x|<1$, we may expand each term by the Binomial Theorem. Thus the given series

$$\begin{array}{llllll} =x-2x^2+3x^3-4x^4+\ldots & \ldots & \ldots & \ldots \\ +x^2-2x^4+3x^6-4x^8+ & \ldots & \ldots & \ldots \\ +x^3-2x^6+3x^9-4x^{12}+ & \ldots & \ldots & \ldots \\ \ldots\quad\ldots\quad\ldots & \ldots & \ldots & \ldots \\ +x^p-2x^{2p}+3x^{3p}-4x^{4p}+\ldots & \ldots & \ldots & \ldots \\ \ldots\quad\ldots\quad\ldots & \ldots & \ldots & \ldots \end{array}$$

Taking the sums of the terms in the various columns, this is

$$\frac{x}{1-x}-\frac{2x^2}{1-x^2}+\frac{3x^3}{1-x^3}-\quad\ldots\quad\ldots\quad\ldots$$

If $d_1, d_2 \ldots\ldots d_k$ are the divisors of μ, the term Ax^{μ} will occur in those of the above rows which begin with $x^{d_1}, x^{d_2}, x^{d_3}\ldots\ldots\ldots$ In each case, $A=\pm\mu/d_r=\pm e_r$, e_r being the complementary divisor of d_r, and the $+$ or $-$ sign being taken according as e_r is odd or even.

Hence $\psi(\mu)=$the excess of the odd over the even divisors.

Question 404.

(A. C. L. Wilkinson, M.A., F.R.A.S.) :—In a tetrahedron, for which the products of the pairs of opposite edges are equal, prove that any plane parallel to the tangent plane to the circumscribing sphere at one of the vertices, cuts the three edges through that vertex in an equilateral triangle.

Solution by J. C. Swaminarayan, M.A. and V. B. Naik, M.A.

Let OABC be the given tetrahedron.

The tangent-plane at O to the circumscribing sphere is taken as the plane of x, y.

The sphere will be represented by

$$x^2+y^2+z^2=2\rho z, \text{ where } \rho \text{ is the radius.}$$

Let A,B,C be $(x_1y_1z_1)$, $(x_2y_2z_2)$, $(x_3y_3z_3)$ respectively.

By hypotheis

$$BC^2\Sigma x_1^2=CA^2\Sigma x_2^2=AB^2\Sigma x_3^2$$

$$\therefore \quad \frac{BC^2}{z_2z_3}=\frac{CA^2}{z_3z_1}=\frac{AB^2}{z_1z_2} \quad\ldots\quad\ldots \quad (1)$$

Let $z=k$, cut OA, OB, OC in A′,B′,C′.

Now $$BC^2=\Sigma x_2^2+\Sigma x_3^2-2\Sigma x_2x_3$$

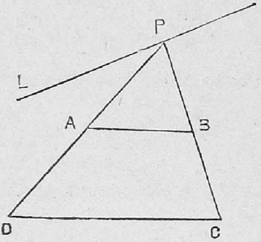
∴ $$2\Sigma x_2x_3=2\rho z_2+2\rho z_3-BC^2.$$

∴ $$B'C'^2=k^2\left[\frac{\Sigma x_2^2}{z_2}+\frac{\Sigma x_3^2}{z_3}-\frac{2\Sigma x_2x_3}{z_2z_3}\right]$$
$$=k^2\left[\frac{2\rho}{z_2}+\frac{2\rho}{z_3}-\frac{2\rho z_2+2\rho z_3-BC}{z_2z_3}\right]=\frac{k^2BC^2}{z_3z_3}$$
$$=C'A'^2=A'B'^2,\ \text{by}(1)$$

Additional solution by N. B. Pendse.

Question 405.

(M. V. Arunachala Sastry, M.A.) :—Construct a trapezium ABCD, which can have a circle inscribed in it, given one of the parallel sides AB in position and magnitude, the length of the parallel side, and a line L on which the oblique sides intersect.



Solution by M. Bhimasena Rao and others.

Let the oblique sides AD, BC meet in P.

Since CD is parallel to AB,

$$PA : PD=PB : PC=AB : CD$$

∴ $$PA : AD=PB : BC=AB : CD-AB$$

∴ $$PA+PB : AD+BC=AB : CD-AB.$$

Now $$AD+BC=AB+CD$$

∴ $$PA+PB=\frac{AB(AB+DC)}{(CD-AB)},\ \text{which is fixed.}$$

Therefore P moves on an ellipse whose foci are at A and B, and major axis known. The question therefore reduces to finding the points of intersection of a given line and an ellipse whose foci and major axis are given.

Question 407.

(D. D. KAPADIA, M.A., B.Sc.):—The polars of the point (x,y,z) with respects to the three conics $S_1=0$, $S_2=0$, $S_3=0$, are concurrent : prove that the point (x, y, z), as well as the point of concurrence, lie on the Jacobian

$$\sum \pm \frac{\partial S_1}{\partial x}, \frac{\partial S_2}{\partial y}. \frac{\partial S_3}{\partial z}=0.$$

Solution by J. C. Swaminaraynn, M.A., M. R. Sadasiva Aiyar, M.A., B.E.; V. B. Naik, M.A. and N.B. Pendse.

Suppose the polars of P (x,y,z) with respect to $S_1=0$, $S_2=0$, $S_3=0$ are concurrent at Q. Then the polars of Q will be concurrent at P.

The equation to the polars are

$$X\frac{\partial S_1}{\partial x}+Y\frac{\partial S_1}{\partial y}+Z\frac{\partial S_1}{\partial z}=0,$$

$$X\frac{\partial S_2}{\partial x}+Y\frac{\partial S_2}{\partial y}+Z\frac{\partial S_2}{\partial z}=0,$$

$$X\frac{\partial S_3}{\partial x}+Y\frac{\partial S_3}{\partial y}+Z\frac{\partial S_3}{\partial z}=0;$$

whence the equation to the locus of (x,y,z) is the vanishing of the Jacobian

$$\frac{\partial(S_1, S_2, S_3)}{\partial(x, y, z)}.$$

As the polars at Q are concurrent, Q also lies on the same locus.

Question 408.

(J. C. SWAMINARAYAN, M.A.):—A regular polygon $A_1A_2A_3\ldots A_m$ of m sides and a regular polygon $B_1B_2B_3\ldots B_n$ of n sides are inscribed in a circle of radius r. Shew that the continued product of all the chords AB is numerically equal to $2r^{mn}\sin(m+n)\theta$, where 2θ is the angle subtended by one of these chords at the centre of the circle. (Extension of Q. 389 of Mr. D. D. Kapadia, M.A., B. Sc.)

Solution by K. J. Sanjana M. A., V. B. Naik, M. A., T. P. Trivedi M. A., LL. B., and N. B. Pendse.

The result given is incorrect, and the proper wording would be— "shew that the continued product of all the chords AB is numerically equal to $r^{mn}(2\sin\nu\theta)^{\mu}$, where μ is the greatest common measure and ν the least common multiple of m and n, and 2θ is the angle subtended at the centre by one of these chords."

Let any one of the angles AOB, say A_1OB_1, be $2\,\Theta$; then $A_1B_1 = 2\,r\sin\Theta$; $A_1B_2 = 2\,r\sin\left(\Theta+\frac{\pi}{n}\right)$, as $\angle A_1OB_2 = 2\,\Theta + 2\,\pi/n$; and so on.

Hence.

$$A_1B_1 \cdot A_1B_2 \ldots A_1B_n = 2^n \cdot r^n \cdot \sin\Theta\,\sin\left(\Theta+\frac{\pi}{n}\right)\ldots\sin\left(\Theta+\frac{n-1}{n}\pi\right)$$

$$= 2\,r^n\sin n\,\Theta, \text{ by a well known result.}$$

Similarly, $\quad A_2B_1 \cdot A_2B_2 \ldots A_2B^n = 2r^n \sin\left((n\Theta - \frac{n}{m}\pi.\right)$

So also, $A_3B_1 \cdot A_3B_2 \ldots, \ldots,\ A_nB_1 \cdot A_nB_3 \ldots A_nB_2 \ldots$ may all be found.

Thus the continued product of all the shords AB is

$$2^m\,r^{mn}.\sin n\,\Theta \sin\left(n\Theta - \frac{n}{m}\pi\right)\sin\left(n\Theta - \frac{m-1}{m}n\pi.\right).$$

Now, let $m = \mu p$, $n = \mu q$, where p and q are prime to each other, adn put $\nu = \mu pq$. In the series of quotients.

$$\frac{n}{m}, \frac{2n}{m}, \frac{3n}{m}, \ldots\ldots\ldots \frac{m-1}{m}n,$$

the remainders will be the same as those of

$$\frac{q}{p}, \frac{2q}{p}, \frac{3q}{p}, \ldots\ldots\ldots \frac{(m-1)\,q}{p}.$$

Now q and p are prime to each other, so that only p of these remainders (including zero) will be distinct and will be repeated μ times over to make up the m terms of the series.

Thus the product of the chords AB is

$$2^m\,r^{mn}\left\{\sin n\Theta \sin\left(n\,\Theta - \frac{\pi}{p}\right)\sin\left(n\,\Theta - \frac{2\pi}{p}\right)\ldots\left(\sin n\,\Theta - \frac{p-1}{p}\pi\right)\right\}^{\mu}$$

Adding π to each angle and taking only the numerical values of the since, we get

$$2^m r^{mn}\left\{\sin n\Theta \sin\left(n\Theta + \frac{\pi}{p}\right)\sin\left(n\Theta + \frac{2\pi}{p}\right)\ldots\sin\left(n\Theta + \frac{p-1}{p}\pi\right)\right\}^{\mu};$$

$$= r^{mn}\left\{2\sin pn\,\Theta\right\}^{\mu}$$

$$= r^{mn}\,(2\sin\nu\Theta)^{\mu}.$$

When m and n are prime to each other $\mu = 1$, $\nu = mn$, and the resuls is $2\,r^{mn}\sin mn\,\Theta$. If any point A_k coincides with a point B_k, Θ it $k\pi\left(\frac{1}{m} - \frac{1}{n}\right)$ and $\sin\nu\,\Theta$ vanishes, as was to be expected.

QUESTION FOR SOLUTION.

438. (A. C. L. Wilkinson, M.A., F.R.A.S.) :—If through a point P, in the plane of a triangle ABC, straight lines are drawn parallel to the sides, shew that the six points of intersection with the sides lie on a conic which is an ellipse, parabola or hyperbola according as P lies within, on, or outside the maximum inscribed ellipse of ABC. Shew also that it is two straight lines, if P lies on the minimum ellipse circumscribing ABC. (Suggested by Question No. 429).

439. (S. P. Singaravelu Moodeliar) :—The circle of curvature at P to the Folium of Descartes passes through the node and cuts the curve again at Q. Shew that the envelope of PQ is a rectangular hyperbola. Also find the locus of intersection of the tangents at P and Q to the Folium.

440. (P. V. Seshu Aiyar) :—Show that

$$(1)\quad \int_0^\infty \frac{\sinh .px}{\sinh .qx} \cos rx\, dx = \frac{\pi}{2q} \frac{\sin\frac{p\pi}{q}}{\cos\frac{p\pi}{q} + \cos \mathrm{h}\frac{r\pi}{q}}$$

$$(2)\quad \int_0^\infty \frac{\cosh .px}{\sinh .qx} \sin rx\, dx = \frac{\pi}{2q} \frac{\sin \mathrm{h}\frac{r\pi}{q}}{\cos\frac{p\pi}{q} + \cosh\frac{r\pi}{q}}$$

441. (S. Ramanujan) :—Shew that

$$(6a^2-4ab+4b^2)^3=(3a^2+5ab-5b^2)^3+(4a^2-4ab+6b^2)^3 +(5a^2-5ab-3b^2)^3,$$

and find other quadratic expressions satisfying similar relations.

442. (K. J. Sanjana, M.A.) :—Obtain a general solution of the Diophantine equation

$$\frac{ka^2x^3+}{kb^2x+d}=\text{square}.$$

Find four solutions in rational numbers of the equation

$$\frac{75x^3-125}{27x-125}=\text{square}.$$

443. (K. APPUKUTTAN ERADY, M.A.) :—Points are taken on the principal normals to a curve in space at distances from the curve equal to c times the circular curvature of the curve. Prove that

$$\left(\frac{ds'}{ds}\right)^2=\left(1-\frac{c^2}{\rho^2}\right)+\frac{c^2}{\rho^2}\left\{\frac{1}{\sigma^2}+\frac{1}{\rho^2}\left(\frac{d\rho}{ds}\right)^2\right\}$$

where s' refers to the locus of the points in question.

444. (J. C. SWAMINARAYAN, M.A.) :—On the base BC of a triangle ABC equilateral triangles BPC and BQC are described. Shew that the bisectors of the angle PAQ are parallel to the axes of the maximum inscribed ellipse of the triangle ABC.

445. (D. D. KAPADIA, M.A., B.Sc.) :—The equation of a family of curves is $f(x,y,a)=0$, where a is the variable parameter. If the *envelope* of this family of curves has a contact of the second order with the curve, prove that at the point of contact

$$\frac{\partial f}{\partial x}\cdot\frac{\partial^2 f}{\partial y.\partial a}-\frac{\partial f}{\partial y}\cdot\frac{\partial^2 f}{\partial x.\partial a}=0.$$

446. (A. C. L. WILKINSON, M.A., F.R.A.S.) :—In Mr. Swaminarayan's note on "A generalized form of Clairaut's Equation," I.M.J., Vol. 4, No. 6, discuss the cases : $b=1$, and $a+b=1$. Illustrate by solving the equation :

$$(y-xp)^2=4py.$$